

Contents

Introduction	1
CHAPTER I	
Origins of Formal Structure	6
1. The Natural Numbers	7
2. Infinite Sets	10
3. Permutations	11
4. Time and Order	13
5. Space and Motion	16
6. Symmetry	19
7. Transformation Groups	21
8. Groups	22
9. Boolean Algebra	26
10. Calculus, Continuity, and Topology	29
11. Human Activity and Ideas	34
12. Mathematical Activities	36
13. Axiomatic Structure	40
CHAPTER II	
From Whole Numbers to Rational Numbers	42
1. Properties of Natural Numbers	42
2. The Peano Postulates	43
3. Natural Numbers Described by Recursion	47
4. Number Theory	48
5. Integers	50
6. Rational Numbers	51
7. Congruence	52
8. Cardinal Numbers	54
9. Ordinal Numbers	56
10. What Are Numbers?	58

CHAPTER III	
Geometry	61
1. Spatial Activities	61
2. Proofs without Figures	63
3. The Parallel Axiom	67
4. Hyperbolic Geometry	70
5. Elliptic Geometry	73
6. Geometric Magnitude	75
7. Geometry by Motion	76
8. Orientation	82
9. Groups in Geometry	85
10. Geometry by Groups	87
11. Solid Geometry	89
12. Is Geometry a Science?	91
 CHAPTER IV	
Real Numbers	93
1. Measures of Magnitude	93
2. Magnitude as a Geometric Measure	94
3. Manipulations of Magnitudes	97
4. Comparison of Magnitudes	98
5. Axioms for the Reals	102
6. Arithmetic Construction of the Reals	105
7. Vector Geometry	107
8. Analytic Geometry	109
9. Trigonometry	110
10. Complex Numbers	114
11. Stereographic Projection and Infinity	116
12. Are Imaginary Numbers Real?	118
13. Abstract Algebra Revealed	119
14. The Quaternions—and Beyond	120
15. Summary	121
 CHAPTER V	
Functions, Transformations, and Groups	123
1. Types of Functions	123
2. Maps	125
3. What Is a Function?	126
4. Functions as Sets of Pairs	128
5. Transformation Groups	133
6. Groups	135
7. Galois Theory	138
8. Constructions of Groups	142
9. Simple Groups	146
10. Summary: Ideas of Image and Composition	147

CHAPTER VI

Concepts of Calculus	150
1. Origins	150
2. Integration	152
3. Derivatives	154
4. The Fundamental Theorem of the Integral Calculus	155
5. Kepler's Laws and Newton's Laws	158
6. Differential Equations	161
7. Foundations of Calculus	162
8. Approximations and Taylor's Series	167
9. Partial Derivatives	168
10. Differential Forms	173
11. Calculus Becomes Analysis	178
12. Interconnections of the Concepts	183

CHAPTER VII

Linear Algebra	185
1. Sources of Linearity	185
2. Transformations versus Matrices	188
3. Eigenvalues	191
4. Dual Spaces	193
5. Inner Product Spaces	196
6. Orthogonal Matrices	198
7. Adjoints	200
8. The Principal Axis Theorem	202
9. Bilinearity and Tensor Products	204
10. Collapse by Quotients	208
11. Exterior Algebra and Differential Forms	210
12. Similarity and Sums	213
13. Summary	218

CHAPTER VIII

Forms of Space	219
1. Curvature	219
2. Gaussian Curvature for Surfaces	222
3. Arc Length and Intrinsic Geometry	226
4. Many-Valued Functions and Riemann Surfaces	228
5. Examples of Manifolds	233
6. Intrinsic Surfaces and Topological Spaces	236
7. Manifolds	239
8. Smooth Manifolds	244
9. Paths and Quantities	247
10. Riemann Metrics	251
11. Sheaves	252
12. What Is Geometry?	256

CHAPTER IX	
Mechanics	259
1. Kepler's Laws	259
2. Momentum, Work, and Energy	264
3. Lagrange's Equations	267
4. Velocities and Tangent Bundles	274
5. Mechanics in Mathematics	277
6. Hamilton's Principle	278
7. Hamilton's Equations	282
8. Tricks versus Ideas	287
9. The Principal Function	289
10. The Hamilton–Jacobi Equation	292
11. The Spinning Top	295
12. The Form of Mechanics	301
13. Quantum Mechanics	303
 CHAPTER X	
Complex Analysis and Topology	307
1. Functions of a Complex Variable	307
2. Pathological Functions	310
3. Complex Derivatives	312
4. Complex Integration	317
5. Paths in the Plane	322
6. The Cauchy Theorem	328
7. Uniform Convergence	333
8. Power Series	336
9. The Cauchy Integral Formula	338
10. Singularities	341
11. Riemann Surfaces	344
12. Germs and Sheaves	351
13. Analysis, Geometry, and Topology	356
 CHAPTER XI	
Sets, Logic, and Categories	358
1. The Hierarchy of Sets	359
2. Axiomatic Set Theory	362
3. The Propositional Calculus	368
4. First Order Language	370
5. The Predicate Calculus	373
6. Precision and Understanding	377
7. Gödel Incompleteness Theorems	379
8. Independence Results	383
9. Categories and Functions	386
10. Natural Transformations	390
11. Universals	392
12. Axioms on Functions	398
13. Intuitionistic Logic	402

14. Independence by Means of Sheaves	404
15. Foundation or Organization?	406
 CHAPTER XII	
The Mathematical Network	409
1. The Formal	410
2. Ideas	415
3. The Network	417
4. Subjects, Specialties, and Subdivisions	422
5. Problems	428
6. Understanding Mathematics	431
7. Generalization and Abstraction	434
8. Novelty	438
9. Is Mathematics True?	440
10. Platonism	447
11. Preferred Directions for Research	449
12. Summary	453
 Bibliography	 457
 List of Symbols	 461
 Index	 463